

# OHSAWA-TAKEGOSHI EXTENSION THEOREM FOR COMPACT KÄHLER MANIFOLDS AND APPLICATIONS

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**ABSTRACT.** Our main goal in this article is to prove a new extension theorem for sections of the canonical bundle of a weakly pseudoconvex Kähler manifold with values in a line bundle endowed with a possibly singular metric. We also give some applications of our result.

## 1. INTRODUCTION

The  $L^2$  extension theorem by Ohsawa-Takegoshi is a tool of fundamental importance in algebraic and analytic geometry. After the crucial contribution of [OT87], [Ohs88], this result has been generalized by many authors in various contexts, including [Man93], [Dem01], [Ber05], [Che11], [Li12], [ZGZ12], [Bł13], [GZ13a].

In this article we treat yet another version of the extension theorem in the context of Kähler manifolds. We first state a consequence of our main result; we remark that a version of it was conjectured by Y.-T. Siu in the framework of his work on the invariance of plurigena.

**Theorem 1.1.** *Let  $X$  be a Kähler manifold and  $\text{pr} : X \rightarrow \Delta$  be a proper holomorphic map to the ball  $\Delta \subset \mathbb{C}^1$  centered at 0 of radius  $R$ . Let  $(L, h)$  be a holomorphic line bundle over  $X$  equipped with a hermitian metric (maybe singular)  $h = h_0 e^{-\varphi_L}$  such that  $i\Theta_h(L) \geq 0$  in the sense of currents, where  $h_0$  is a smooth hermitian metric and  $\varphi_L$  is a quasi-psh function over  $X$ . We suppose that  $X_0 := \text{pr}^{-1}(0)$  is smooth of codimension 1, and that the restriction of  $h$  to  $X_0$  is not identically  $\infty$ .*

*Let  $f \in H^0(X_0, K_{X_0} \otimes L)$  be a holomorphic section in the multiplier ideal defined by the restriction of  $h$  to  $X_0$ . Then there exists a section  $F \in H^0(X, K_X \otimes L)$  whose restriction to  $X_0$  is equal to  $f$ , and such that the following optimal estimate holds*

$$(1) \quad \frac{1}{\pi R^2} \int_X |F|^2 e^{-\varphi_L} \leq \int_{X_0} |f|^2 e^{-\varphi_L}.$$

In order to keep the notations simple, in this article we will systematically use the symbol  $|\cdot|$  to denote the norms of sections of line/vector bundles, provided that the metrics we are using are clearly defined from the context.

If the manifold  $X$  is isomorphic to the product  $X_0 \times \Delta$  and if the line bundle  $L$  is trivial, then it is clear how to construct  $F$ . If not, the existence of an extension verifying the estimate above is quite subtle, and it has many important applications. The result above is a generalization of [Bł13], [GZ13a]; the new input here is that we allow the metric  $h$  of  $L$  to be singular, while the ambient manifold is only assumed to be Kähler. This general context leads to rather severe difficulties,

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mainly due to the loss of positivity in the process of regularizing the metric  $h$  which adds to the intricate relationship between the several parameters involved in the proof.

Before stating the main result of this paper in its most general form and explaining the main ideas in the proof, we note the following consequence of Theorem 1.1. It is a generalization of [BP10, Thm 0.1] to arbitrary compact Kähler families, which follows from our main theorem by using the same arguments as in [GZ13a, Cor 3.7].

**Theorem 1.2.** *Let  $p : X \rightarrow Y$  be a fibration between two compact Kähler manifolds. Let  $L \rightarrow X$  be a line bundle endowed with a metric (maybe singular)  $\varphi_L$  such that  $i\Theta_{\varphi_L}(L) \geq 0$ . Suppose that there exists a generic point  $z \in Y$  and a section  $u \in H^0(X_z, mK_{X/Y} + L)$  such that*

$$\int_{X_z} |u|^{\frac{2}{m}} e^{-\frac{\varphi_L}{m}} < +\infty.$$

*Then the line bundle  $mK_{X/Y} + L$  admits a metric with positive curvature current. Moreover, this metric equals to the fiberwise  $m$ -Bergman kernel metric on the generic fibers of  $p$ .*

We note that the original proof of the theorem above in the projective case does not go through in the Kähler case. This is due to the fact that in [BP10, Thm 0.1] the authors are using in an essential manner the existence of Zariski dense open subsets of  $X$ .

We will state next our general version of Theorem 1.1; prior to this, we introduce some auxiliary weights, following [Blo13], [GZ13a].

**Notations 1.** (i): Given  $\delta > 0$  and  $A \in \mathbb{R}$ , let  $c_A(t)$  be a positive smooth function on  $(-A, +\infty)$  such that  $\int_{-A}^{+\infty} c_A(t) e^{-t} dt < +\infty$ . Set

$$u(t) = -\ln\left(\frac{c_A(-A)e^A}{\delta}\right) + \int_{-A}^t c_A(t_1) e^{-t_1} dt_1,$$

and

$$s(t) = \frac{\int_{-A}^t e^{-u(t_1)} dt_1 + \frac{c_A(-A)e^A}{\delta^2}}{e^{-u(t)}}.$$

Then  $u(t)$  and  $s(t)$  satisfy the ODE equations:

$$(2) \quad \left(s(t) + \frac{(s'(t))^2}{u''(t)s(t) - s''(t)}\right) e^{u(t)-t} = \frac{1}{c_A(t)}$$

and

$$(3) \quad s'(t) - s(t)u'(t) = 1.$$

We suppose moreover that

$$(4) \quad e^{-u(t)} \geq c_A(t)s(t) \cdot e^{-t} \quad \text{for every } t \in (-A, +\infty).$$

(ii): The weight  $|\Lambda^r(ds)|^2$  is defined as the unique function such that

$$\int_Z \frac{G}{|\Lambda^r(dv)|^2} dV_{Z,\omega} = \lim_{m \rightarrow +\infty} \int_{-m-1 \leq \ln|v|_{h_E}^{2r} \leq -m} \frac{G}{|v|_{h_E}^{2r}} dV_{X,\omega} \quad \text{for every } G \in C^\infty(X).$$

**Remark 2.** If  $c_A(t) \cdot e^{-t}$  is decreasing, then (4) is automatically satisfied. Moreover, by the construction of  $u(t), s(t)$ , we know that

$$(5) \quad \lim_{t \rightarrow +\infty} u(t) = -\ln\left(\frac{c_A(-A)e^A}{\delta}\right) + \int_{-A}^{+\infty} c_A(t_1)e^{-t_1} dt_1 < +\infty$$

and

$$(6) \quad |s(t)| \leq C_1|t| + C_2$$

for two constants  $C_1, C_2$  independent of  $t$ .

In this set-up, the main result of the present paper in its complete form states as follows.

**Theorem 1.3.** *Let  $(X, \omega)$  be a weakly pseudoconvex  $n$ -dimensional Kähler manifold and  $E$  be a vector bundle of rank  $r$  endowed with a smooth metric  $h_E$ . Let  $Z \subset X$  be the zero locus of  $v \in H^0(X, E)$ . We assume that  $Z$  is smooth of codimension  $r$  and  $|v|_{h_E}^{2r} \leq e^A$  for some  $A \in \mathbb{R}$ . Set  $\Psi(z) := \ln |v|_{h_E}^{2r}$ .*

*Let  $L$  be a line bundle on  $X$  equipped with a singular metric  $h := e^{-\varphi} h_0$  such that  $i\Theta_h(L) \geq \gamma$  for some continuous  $(1, 1)$ -form  $\gamma$ . We assume that there exists a sequence of analytic approximations  $\{\varphi_k\}_{k=1}^\infty$  of  $\varphi$  such that*

$$(7) \quad i\Theta_{\varphi_k}(L) \geq \gamma - \delta_k \omega$$

*for a sequence  $\delta_k \rightarrow 0$ <sup>1</sup>. We suppose that there exists a continuous function  $a(t)$  on  $(-A, +\infty]$ , such that  $0 < a(t) \leq s(t)$  and*

$$(8) \quad a(-\Psi)(\gamma + id'd''\Psi) + id'd''\Psi \geq 0.$$

*Then for every  $f \in H^0(Z, K_X \otimes L \otimes \mathcal{I}(\varphi|_Z))$ , there exists a section  $F \in H^0(X, K_X \otimes L)$  such that  $F|_Z = f$  and*

$$(9) \quad \int_X c_A(-\Psi) |F|^2 e^{-\varphi} dV_{X, \omega} \leq e^{-\lim_{t \rightarrow +\infty} u(t)} \int_Z \frac{|f|^2 e^{-\varphi}}{|\Lambda^r(dv)|^2} dV_{Z, \omega}.$$

**Remark 3.** *As already pointed out in [GZ13a], by taking  $E = \text{pr}^* \mathcal{O}_\Delta$ ,  $v = \text{pr}^* z$ ,  $A = \ln R^2$ ,  $c_A(t) \equiv 1$  and letting  $\delta \rightarrow +\infty$ , Theorem 1.3 implies Theorem 1.1.*

We comment next a few results at the foundation of Theorem 1.3. The original Ohsawa-Takegoshi extension theorem [OT87] deals with the local case, i.e.  $X$  is a pseudoconvex domain in  $\mathbb{C}^n$ . The potential applications of this type of results in global complex geometry become apparent shortly after the article [OT87] appeared, and to this end it was necessary to rephrase it in the context of manifolds. As far as we are aware, the first global version is due to L. Manivel [Man93]. We quote here a simplified version of his result.

**Theorem 1.4.** [Man93, Thm 2] *Let  $X$  be a  $n$ -dimensional Stein manifold, and  $E$  be a holomorphic vector bundle over  $X$  of rank  $r$  with a smooth metric  $h_E$ . Let  $Y \subset X$  be the zero locus of  $s \in H^0(X, E)$ . We assume that  $Y$  is smooth and of codimension  $r$ . Let  $\Omega$  be a  $(1, 1)$ -closed semi-positive form on  $X$  such that*

$$\Omega \otimes \text{Id}_E \geq i\Theta_{h_E}(E)$$

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<sup>1</sup>If  $X$  is compact, such approximation always exists, cf. [Dem12, Chapter 13]

in the sense of Griffiths. Let  $(L, h_0)$  be a line bundle on  $X$  equipped with a smooth metric  $h_0$ , such that there exists a constant  $\alpha > 0$  satisfying

$$i\Theta_{h_0}(L) \geq \alpha\Omega - \text{rid}'d'' \ln |s|_{h_E}^2.$$

Then for every  $f \in H^0(Y, K_Y \otimes L \otimes (\det E)^{-1})$ , there exists a section  $F \in H^0(X, K_X \otimes L)$  such that  $F|_Y = f \wedge (\wedge^r ds)$  and

$$(10) \quad \int_X \frac{|F|_{h_0}^2}{|s|_{h_E}^{2r-2}(1+|s|_{h_E}^2)^\beta} dV_{X,\omega} \leq C \int_Y |f|_{h_0}^2 dV_{Y,\omega},$$

where  $C$  is a numerical constant depending only on  $r$ ,  $\alpha$  and  $\beta$ .

**Remark 4.** Theorem 1.4 can be easily generalised to the case when  $X$  is a weakly pseudoconvex Kähler manifold and the weight function  $|s|_{h_E}^{2r-2}(1+|s|_{h_E}^2)^\beta$  can be ameliorated by  $|s|_{h_E}^{2r}(\ln |s|_{h_E})^2$ , cf. [Dem12, Thm 12.6].

One of the important limitations of Theorem 1.4 is that the metric  $h_0$  is assumed to be smooth. Indeed this is unfortunate, given that in the usual set-up of algebraic geometry one has to deal with extension problems for canonical forms with values in pseudo-effective line bundles. A famous example is the invariance of plurigeners for projective manifolds ([Siu02]): one needs an extension theorem under the hypothesis that the metric  $h_0$  has arbitrary singularities. We remark that the proof of the extension theorem used in the article mentioned above is confined to the case of projective manifolds. Thus, in order to generalize [Siu02] to compact Kähler manifolds, the first step would be to allow the metric  $h_0$  in Theorem 1.4 to have arbitrary singularities.

Among the very few results in this direction we mention the work of Y. Li. In order to keep the discussion simple, we restrict ourselves to the setup in Theorem 1.1. Let  $\mathcal{I}_+(h) := \lim_{\delta \rightarrow 0^+} \mathcal{I}(h^{1+\delta})$ . Y. Li [Li12] established Theorem 1.1<sup>2</sup> for sections  $f$  which belong to the augmented multiplier ideal sheaf  $\mathcal{I}_+(h)$ . It turns out that recently, Guan, Zhou [GZ13b] and Hiep [Hie14] showed that  $\mathcal{I}_+(h) = \mathcal{I}(h)$ . Thus, the conjunction of these two results establish Theorem 1.1.

In this article, we prove our main theorem 1.3 in a more direct way and without using the result [GZ13b], [Hie14]. Our belief is that the techniques we develop here will be useful in other contexts. We now explain the main ideas of our proof.

The first part of the proof combines the approach of [GZ13a] and [Li12]. For every  $k \in \mathbb{N}$  fixed, by using a local Ohsawa-Takegoshi extension theorem, there exists a smooth extension of  $f$ :  $\tilde{f}_k \in C^\infty(X, K_X \otimes L)$  such that

$$(11) \quad \int_X \frac{|\tilde{f}_k|^2}{|v|^{2r}(\ln |v|)^2} e^{-(1+\sigma_k)\varphi_k} dV_{X,\omega} \leq C_1 \cdot \int_Z \frac{|f|^2 e^{-\varphi_k}}{|\Lambda^r(dv)|^2} dV_{Z,\omega},$$

for some  $\sigma_k \ll 1$  depending on  $k$ .

For every  $m \in \mathbb{R}$  fixed, we define a  $C^1$ -function  $b_m$  on  $\mathbb{R}$  such that

$$b_m(t) = t \text{ for } t \geq -m \quad \text{and} \quad b_m''(t) = \mathbf{1}_{\{-m-1 \leq t \leq -m\}}.$$

Set  $g_{m,k} := D''((1 - b_m' \circ \Psi) \cdot \tilde{f}_k)$ . By combining the  $L^2$  estimates used in [GZ13a] and [Li12], for every  $k \in \mathbb{N}$ , we can find a  $m_k \gg k$ , such that

$$(12) \quad g_{m_k,k} = D''\gamma_{m_k,k} + (\delta_k \cdot \eta_{m_k}(z))^{\frac{1}{2}} \beta_{m_k,k}$$

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<sup>2</sup>[Li12] proved it in a more general setting.

and

$$(13) \quad \int_X c_A(-b_{m_k} \circ \Psi) \cdot |\gamma_{m_k, k}|^2 e^{-\varphi_k} dV_{X, \omega} + C_2 \int_X \frac{|\beta_{m_k, k}|^2 e^{-\varphi_k}}{|v|^{2r}} dV_{X, \omega} \\ \leq e^{-\lim_{t \rightarrow +\infty} u(t)} \int_Z \frac{|f|^2 e^{-\varphi_k}}{|\Lambda^r(dv)|^2} dV_{Z, \omega},$$

where  $\eta_m(z)$  is about  $C_3 \cdot \min\{|\ln|z||, m\}$  near  $Z$ . By passing to a subsequence, it is easy to see that  $(1 - b'_{m_k} \circ \Psi) \cdot \tilde{f}_k - \gamma_{m_k, k}$  converges to a holomorphic section  $F \in H^0(X, K_X \otimes L)$  and  $F$  satisfies the optimal estimate (9). The difficulty is to prove that  $F$  is an extension of  $f$ . The reason is that, since  $\eta_m(z)$  is not uniformly bounded (with respect to  $m$ ), it is difficult to have a uniform  $L^2$  estimate.

In [Li12], under an additional assumption that  $f \in H^0(Z, K_X \otimes L \otimes \mathcal{I}_+(\varphi|_Z))$ , she obtain a uniform  $L^2$  estimate. As a consequence, she can prove the extension theorem with respect to the multiplier ideal sheaf  $\mathcal{I}_+(\varphi)$ . The new observation in our article is that, although  $\eta_m(z)$  is not uniformly bounded, at least locally, we can have a uniform  $L^1$ -estimate. More precisely, for every small open set  $U$  in  $X$ , by using Bochner-Martinelli-Koppelman formula, there exists a  $w_{m_k, k}$  such that

$$\bar{\partial} w_{m_k, k} = (\eta_{m_k}(z))^{\frac{1}{2}} \beta_{m_k, k} \quad \text{on } U$$

and

$$\int_{B_\tau} |w_{m_k, k}| dV_\omega \leq C_4 \cdot \tau^{2r} \quad \text{for every small } \tau > 0,$$

where  $B_\tau$  is a radius  $\tau$  neighborhood of  $Z \cap U$  and  $C_4$  is a constant independent of  $\tau$  and  $k$ . Theorem 1.3 can be thus proved by using some elementary calculus.

**Remark 5.** In the situation of Theorem 1.3, if we take the weight function  $c_A(t) \equiv 1$ , then we have Theorem 1.1. There is another weight function which might be useful. If we take  $c_A(t) = \frac{e^t}{(t+A+c)^2}$  for some constant  $c > 0$ , thanks to Remark 2, (4) is satisfied. Using this weight function, [GZ13a, Thm 3.16] proved an optimal estimate version of Theorem 1.4 and its remark 4. Thanks to Theorem 1.3, we know that [GZ13a, Thm 3.16] is also true for weakly pseudoconvex Kähler manifolds under the approximation assumption (7).

The organization of the article is as follows. In section 2, we first sketch the proof of Theorem 1.3 where we also introduce several important notations and basic estimates. We then give the complete proof of Theorem 1.3. In section 3, we give the proof of Theorem 1.2. In the appendix, for the reader's convenience, we give the proof of a key  $L^2$  estimate, which is essentially the same as [Dem12, Remark 12.5].

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## 2. BASIC NOTATIONS AND THE PROOF OF THEOREM 1.3

For the reader's convenience, we first sketch the proof of Theorem 1.3. In this sketch, we also introduce several notations which will be used freely in the next section.

In the setting of Theorem 1.3, for every  $m \in \mathbb{R}$  fixed, we can define a  $C^1$ -function  $b_m$  on  $\mathbb{R}$  such that

$$b_m(t) = t \text{ for } t \geq -m \quad \text{and} \quad b_m''(t) = \mathbf{1}_{\{-m-1 \leq t \leq -m\}}.$$

Then

$$(14) \quad b_m(t) \geq t \quad \text{and} \quad b_m(t) \geq -m-1 \quad \text{for every } t \in \mathbb{R}^1.$$

Let  $s, u$  be the two functions defined in the introduction. Set  $\chi_m(z) := -b_m \circ \Psi$ ,  $\eta_m(z) := s \circ \chi_m$  and  $\phi_m(z) := u \circ \chi_m$ . Thanks to (5) and (6), we have

$$(15) \quad |\phi_m(z)| \leq C_1$$

and

$$(16) \quad |\eta_m(z)| \leq C_2 |\chi_m(z)| + C_3 \leq C_2 \cdot \min\{2r |\ln |v||, m+1\} + C_3.$$

Set  $\lambda_m(z) := \frac{(s')^2}{u''s-s''} \circ \chi_m$  and  $\tilde{h}_{m,k} = h_k \cdot e^{-\Psi-\phi_m}$ . By (2), we have

$$(17) \quad c_A(\chi_m) \cdot e^{-\chi_m+\phi_m} = (\eta_m + \lambda_m)^{-1}.$$

To prove Theorem 1.3, since the function  $\varphi$  is not necessary smooth, it is very natural take an analytic approximation  $\{\varphi_k\}_{k=1}^\infty$  of  $\varphi$  (cf. [Dem12, Chapter 13]) and use the  $L^2$ -estimate in [GZ13a] for every  $\varphi_k$ . In particular, the curvature term

$$(18) \quad B_{m,k} := [\eta_m(i\Theta_{\tilde{h}_{m,k}}(L)) - id'd''\eta_m - \lambda_m^{-1}id'd''\eta_m \wedge d''\eta_m, \Lambda_\omega]$$

satisfies

$$B_{m,k} \geq (b_m'' \circ \Psi) \cdot [\partial\Psi \wedge \bar{\partial}\Psi, \Lambda_\omega] - \delta_k \eta_m \text{Id}.$$

By using the arguments in [Li12], we can construct an "approximate optimal extension" (cf. (20) and (21)). However, since  $\eta_m$  is about  $C_2 \cdot \min\{|\ln |v||, m\} + C_3$  near  $Z$ , the negativity of  $B_{m,k}$  is not uniformly bounded. This makes trouble when we do the  $L^2$  estimate. In the article of [Li12], she solve this difficulty by assuming that  $f \in H^0(Z, K_X \otimes L \otimes \mathcal{I}_+(\varphi|_Z))$ . We now explain our direct approach, where the construction of the "approximate extensions" follows the arguments in [Li12].

*Sketch of the proof of Theorem 1.3 :* The constants  $C_1, C_2, \dots$  below are all independent of  $k$  and  $m$ . For every  $k \in \mathbb{N}$  fixed, by using a local Ohsawa-Takegoshi extension theorem, there exists a  $C^\infty$  extension of  $f$ :  $\tilde{f}_k \in C^\infty(X, K_X + L)$  such that

$$(19) \quad \int_X \frac{|\tilde{f}_k|^2}{|v|^{2r}(\ln |v|)^2} e^{-(1+\sigma_k)\varphi_k} dV_{X,\omega} \leq C_1 \cdot \int_Z \frac{|f|^2 e^{-\varphi_k}}{|\Lambda^r(dv)|^2} dV_{Z,\omega},$$

for some  $\sigma_k \ll 1$  depending on  $k$ .

Set  $g_{m,k} := D''((1-b_m' \circ \Psi) \cdot \tilde{f}_k)$ . Since  $\varphi_k$  has analytic singularities, by combining the  $L^2$ -estimates in [GZ13a] and [Li12], for every  $k \in \mathbb{N}$ , we can find a  $m_k \gg \delta_k$  such that

$$(20) \quad g_{m_k,k} = D''\gamma_{m_k,k} + (\delta_k \cdot \eta_{m_k}(z))^{\frac{1}{2}} \beta_{m_k,k}$$

and

$$(21) \quad \int_X c_A(-b_{m_k} \circ \Psi) \cdot |\gamma_{m_k, k}|^2 e^{-\varphi_k} dV_{X, \omega} + C_2 \int_X \frac{|\beta_{m_k, k}|^2 e^{-\varphi_k}}{|s|^{2r}} dV_{X, \omega} \\ \leq e^{-\lim_{t \rightarrow +\infty} u(t)} \cdot \int_Z \frac{|f|^2 e^{-\varphi_k}}{|\Lambda^r(ds)|^2} dV_{Z, \omega}.$$

Thanks to (16), by passing to a subsequence,  $(1 - b'_{m_k} \circ \Psi) \cdot \tilde{f}_k - \gamma_{m_k, k}$  converges to a holomorphic section  $F \in H^0(X, K_X \otimes L)$  and  $F$  satisfies the optimal estimate (9). To complete the proof of Theorem 1.3, we rest to check that  $F|_Z = f$ .

Let  $U$  be any small Stein open set of  $X$ . We suppose for simplicity that  $r = 1$  and  $U \cap Z$  is defined by  $z_1 = 0$ . We now prove  $F = f$  on  $U \cap Z$ . Set

$$B_\tau := \{(z_1, \dots, z_n) \mid |z_1| < \tau, (z_2, \dots, z_n) \in U \cap Z\}.$$

Let  $a$  be a positive constant such that  $B_a$  is contained in  $U$ . By using Bochner-Martinelli-Koppelman formula, we can construct a  $w_{m_k, k}$  on  $B_a$ , such that

$$(22) \quad \bar{\partial} w_{m_k, k} = (\eta_{m_k}(z))^{\frac{1}{2}} \beta_{m_k, k} \quad \text{and} \quad \int_{B_\tau} |w_{m_k, k}| dV_\omega \leq C_3 \cdot \tau^2$$

for every  $\tau \leq a$  and  $k \in \mathbb{N}$ .

Once we can construct such  $w_{m_k, k}$ , the function

$$F_k := (1 - b'_{m_k} \circ \Psi) \cdot \tilde{f}_k - \gamma_{m_k, k} - (\delta_k)^{\frac{1}{2}} \cdot w_{m_k, k}.$$

is holomorphic and

$$(23) \quad \lim_{k \rightarrow +\infty} \|F_k - F\|_{L^1(B_{\frac{a}{2}})} = 0.$$

Using (22), (23) and the properties of holomorphic functions, we can finally prove that  $F = f$  on  $U \cap Z$ .  $\square$

We now give the complete proof of Theorem 1.3. The proof is divided by 3 steps. In Step 1 and Step 2, by combining the arguments in [GZ13a] and [Li12], we prove the key  $L^2$  estimate (21). In Step 3, by admitting (22) (which will be proved in Lemma 2.2), we prove that  $F$  is an extension of  $f$ .

*proof of Theorem 1.3.* The constants  $C_1, C_2, \dots$  below are all independent of  $k$ .

**Step 1: construction of smooth extension and parameter functions**

For every  $k \in \mathbb{N}$  fixed, we first construct a smooth section  $\tilde{f}_k \in C^\infty(X, K_X \otimes L)$ , such that

$$(i) \quad \tilde{f}_k|_Z = f.$$

$$(ii) \quad D''(\tilde{f}_k) = 0 \text{ on } Z.$$

$$(iii) \quad \int_X \frac{|\tilde{f}_k|^2}{|v|^{2r} (\ln |v|)^2} \cdot e^{-(1+\sigma_k)\varphi_k} dV_{X, \omega} \leq C_1 \cdot \int_Z \frac{|f|^2 e^{-\varphi_k}}{|\Lambda^r(ds)|^2} dV_{Z, \omega}, \text{ where } \sigma_k > 0 \text{ is a small constant depending on } k.$$

In fact, let  $\{U_i\}$  be a Stein cover of  $X$  and let  $\{\psi_i\}$  be a partition of unity. Since  $\varphi_k$  has analytic singularities, we have  $\mathcal{I}(\varphi_k) = \lim_{\tau \rightarrow 0^+} \mathcal{I}((1 + \tau)\varphi_k)$ . There exists thus a  $\sigma_k > 0$ , such that

$$\int_{U_i \cap Z} |f|^2 e^{-(1+\sigma_k)\varphi_k} dV_{Z, \omega} \leq 2 \int_{U_i \cap Z} |f|^2 e^{-\varphi_k} dV_{Z, \omega}.$$

Applying the local Ohsawa-Takegoshi extension theorem (cf. for example [Dem12, Thm 12.6]) to the weight  $e^{-(1+\sigma_k)\varphi_k}$  on  $U_i$ , we obtain a holomorphic section  $f_{i,k}$  on  $U_i$ , such that

$$\int_{U_i} \frac{|f_{i,k}|^2}{|v|^{2r}(\ln|v|)^2} \cdot e^{-(1+\sigma_k)\varphi_k} dV_{X,\omega} \leq C_2 \cdot \int_{U_i \cap Z} \frac{|f|^2 e^{-\varphi_k}}{|\Lambda^r(ds)|^2} dV_{Z,\omega}.$$

Let  $\tilde{f}_k = \sum_i \psi_i f_{i,k}$ . It is easy to check that  $\tilde{f}_k$  satisfies the above three properties.

**Step 2:  $L^2$  estimate**

Set  $g_{m,k} := \bar{\partial}((1 - b'_m \circ \Psi) \cdot \tilde{f}_k)$  and  $\tilde{h}_{m,k} := h_k \cdot e^{-\Psi - \phi_m}$ , where  $h_k := h_0 \cdot e^{-\varphi_k}$ .

**Claim 1:** There exist  $\gamma_{m,k}$  and  $\beta_{m,k}$  such that

$$(24) \quad \bar{\partial}\gamma_{m,k} + (\delta_k \eta_m)^{\frac{1}{2}} \beta_{m,k} = g_{m,k}$$

and for every  $k \in \mathbb{N}$  fixed,

$$(25) \quad \overline{\lim}_{m \rightarrow +\infty} \left( \int_X \{\gamma_{m,k}, \gamma_{m,k}\}_{\tilde{h}_{m,k}} (\eta_m + \lambda_m)^{-1} + \int_X \{\beta_{m,k}, \beta_{m,k}\}_{\tilde{h}_{m,k}} \right) \\ \leq \lim_{m \rightarrow +\infty} \int_X (b''_m \circ \Psi) \{\tilde{f}_k, \tilde{f}_k\}_{\tilde{h}_{m,k}}.$$

The proof of the claim combines a regularization process (since  $b_m$  is only  $C^1$ ) and a standard  $L^2$ -estimate. We postpone the proof of the claim in Lemma 2.1 and first finish the proof of the theorem.

We first estimate the right-hand side of (25). By construction, we have

$$(26) \quad \lim_{m \rightarrow +\infty} \int_X (b''_m \circ \Psi) \{\tilde{f}_k, \tilde{f}_k\}_{\tilde{h}_{m,k}} \\ = e^{-\lim_{t \rightarrow +\infty} u(t)} \cdot \lim_{m \rightarrow +\infty} \int_{-m-1 \leq \Psi \leq -m} \{\tilde{f}_k, \tilde{f}_k\}_{h_k} \cdot e^{-\Psi} \\ = e^{-\lim_{t \rightarrow +\infty} u(t)} \cdot \int_Z \frac{|f|^2 e^{-\varphi_k}}{|\Lambda^r(ds)|^2} dV_{Z,\omega}.$$

We need also estimate  $\int_X c_A(-b_m \circ \Psi) \cdot \{\gamma_{m,k}, \gamma_{m,k}\}_{h_k}$ . By (14) and (17), we have

$$c_A(-b_m \circ \Psi) \cdot e^{\Psi + \phi_m} = c_A(\chi_m) \cdot e^{\Psi + \phi_m} \leq (\eta_m + \lambda_m)^{-1}.$$

Therefore

$$(27) \quad \int_X c_A(-b_m \circ \Psi) \cdot \{\gamma_{m,k}, \gamma_{m,k}\}_{h_k} \leq \int_X \{\gamma_{m,k}, \gamma_{m,k}\}_{\tilde{h}_{m,k}} (\eta_m + \lambda_m)^{-1}.$$

Combining with (25) and (26), for every  $k \in \mathbb{N}$ , we can take a  $m_k \gg k$  such that

$$(28) \quad \overline{\lim}_{k \rightarrow +\infty} \int_X c_A(-b_{m_k} \circ \Psi) \{\gamma_{m_k,k}, \gamma_{m_k,k}\}_{h_k} \leq e^{-\lim_{t \rightarrow +\infty} u(t)} \cdot \int_Z \frac{|f|^2 e^{-\varphi_k}}{|\Lambda^r(ds)|^2} dV_{Z,\omega}.$$

By passing to a subsequence, we can thus assume that the sequence

$$\{\gamma_{m_k,k} - (1 - b'_{m_k} \circ \Psi) \tilde{f}_k\}_{k=1}^{+\infty}$$

converges weakly (in the weak  $L^2$ -sense) to a section  $F \in L^2(X, K_X \otimes L)$ .

Thanks to (15) and (16), (25) implies that

$$(29) \quad \int_X \{\beta_{m,k}, \gamma_{m,k}\}_{h_k} |\eta_m|^2 \leq C_3$$



for a uniform constant  $C_3$ . Since  $\delta_k$  tends to 0, (24) and (29) imply that  $F$  is a holomorphic section.

We now check that  $F$  satisfies the optimal estimate (9). For every  $k_0 \in \mathbb{N}$  fixed, since  $\varphi_k$  is a decreasing sequence, we have

$$(30) \quad \int_X c_A(-b_{m_k} \circ \Psi) \{\gamma_{m_k, k}, \gamma_{m_k, k}\}_{h_{k_0}} \leq \int_X c_A(-b_{m_k} \circ \Psi) \{\gamma_{m_k, k}, \gamma_{m_k, k}\}_{h_k}$$

for every  $k > k_0$ . By Fatou's lemma and (28), we have

$$\int_X c_A(-\Psi) \{F, F\}_{h_{k_0}}^2 dV_{X, \omega} \leq e^{-\lim_{t \rightarrow +\infty} u(t)} \int_Z \frac{|f|^2 e^{-\varphi}}{|\Lambda^r(ds)|^2} dV_{Z, \omega},$$

By letting  $k_0 \rightarrow +\infty$ , (25) is proved.

We would like to prove in the next step that  $F|_Z = f$ . It is at this point that our main contribution to the subject appears.

**Step 3:  $F$  is an extension of  $f$ , final conclusion**

Let  $\{U_i\}$  be a Stein cover of  $X$ . To finish the proof of the theorem, we need to prove that  $F|_{U_i \cap Z} = f$  for every  $i$ . We suppose for simplicity that the coordinate of  $U_i$  is given by  $\{z_1, z_2, \dots, z_n\}$  and  $Z \cap U_i$  is defined by  $z_1 = \dots = z_r = 0$ . Set

$$B_\tau := \{(z_1, \dots, z_n) \mid \sup\{|z_1|, \dots, |z_r|\} \leq \tau, (z_{r+1}, \dots, z_n) \in Z \cap U_i\}.$$

Let  $a$  be a strictly positive number such that  $B_a$  is contained in  $U_i$ . We note that the norms  $\|\cdot\|$  and  $|\cdot|$  in this step are the standard norms on  $\mathbb{C}^n$  and  $\mathbb{C}$ , respectively.

**Claim 2:** There exists a  $L^1$  function  $w_{m_k, k}$  on  $B_{\frac{3a}{4}}$  such that

$$(31) \quad \bar{\partial} w_{m_k, k} = (\eta_{m_k}(z))^{\frac{1}{2}} \beta_{m_k, k} \quad \text{and} \quad \int_{B_\tau} |w_{m_k, k}| dV_X \leq C_4 \cdot \tau^{2r}$$

for every  $\tau \leq \frac{3a}{4}$  and  $k \in \mathbb{N}$ . We postpone the construction of  $w_{m_k, k}$  in Lemma 2.2 and first finish the proof of the theorem.

By construction, the function

$$F_k := (1 - b'_m \circ \Psi) \cdot \tilde{f}_k - \gamma_{m_k, k} - (\delta_k)^{\frac{1}{2}} \cdot w_{m_k, k}$$

is holomorphic and  $F_k \rightarrow F$  in  $L^1(B_{\frac{3a}{4}})$ . Since  $F$  and  $F_k$  are holomorphic, we have

$$(32) \quad \lim_{k \rightarrow +\infty} \|F_k - F\|_{L^1(B_{\frac{a}{2}})} = 0,$$

and the key uniform estimate

$$(33) \quad \frac{1}{\tau^{2r}} \int_{B_\tau} |F_k - F| dV_\omega \leq C_5 \|F_k - F\|_{L^1(B_{\frac{a}{2}})} \quad \text{for every } \tau < \frac{1}{4}a.$$

Set  $\tilde{f}(z_1, z_2, \dots, z_n) := f(z_{r+1}, \dots, z_n)$ . We first prove that, for every  $k \in \mathbb{N}$  fixed

$$(34) \quad \lim_{\tau \rightarrow 0} \frac{1}{\tau^{2r}} \int_{B_\tau} |F_k - \tilde{f}| dV_\omega \leq C_4 (\delta_k)^{\frac{1}{2}}.$$

In fact, by the construction of  $\tilde{f}_k$  and  $w_{m_k, k}$ , we have

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau^{2r}} \int_{B_\tau} |(1 - b'_m \circ \Psi) \cdot \tilde{f}_k - (\delta_k)^{\frac{1}{2}} w_{m_k, k} - \tilde{f}| dV_\omega \leq C_4 (\delta_k)^{\frac{1}{2}}$$

To prove (34), it is thus sufficient to prove:

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau^{2r}} \int_{B_\tau} |\gamma_{m_k, k}| dV_\omega = 0.$$

By Cauchy inequality,

$$(35) \quad \int_{B_\tau} |\gamma_{m_k, k}| dV_\omega \leq \left( \int_{B_\tau} e^{\Psi + \phi_{m_k}} (\eta_{m_k} + \lambda_{m_k}) dV_\omega \right)^{\frac{1}{2}} \cdot \left( \int_{B_\tau} \frac{|\gamma_{m_k, k}|_{\tilde{h}_{m_k, k}}^2}{(\eta_{m_k} + \lambda_{m_k})} dV_\omega \right)^{\frac{1}{2}}$$

Using (14) and (15), we know that<sup>3</sup>

$$\left( \int_{B_\tau} e^{\Psi + \phi_{m_k}} (\eta_{m_k} + \lambda_{m_k}) dV_\omega \right)^{\frac{1}{2}} \leq C(m_k) \cdot \tau^{2r},$$

where  $C(m_k)$  is a constant depending on  $m_k$  but not depending on  $\tau$ . Moreover,

for every  $k \in \mathbb{N}$  fixed, since  $\int_X \frac{|\gamma_{m_k, k}|_{\tilde{h}_{m_k, k}}^2}{(\eta_{m_k} + \lambda_{m_k})} dV_\omega < +\infty$ , we have

$$(36) \quad \lim_{\tau \rightarrow 0} \int_{B_\tau} \frac{|\gamma_{m_k, k}|_{\tilde{h}_{m_k, k}}^2}{(\eta_{m_k} + \lambda_{m_k})} dV_\omega = 0.$$

Therefore,

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau^{2r}} \int_{B_\tau} |\gamma_{m_k, k}| dV_\omega = 0 \quad \text{for every } k \text{ fixed,}$$

and (34) is proved.

We now prove that  $F|_{Z \cap U_i} = f$ . We first note that, thanks to (34), there exists a  $\tau_k > 0$  depending on  $k$ , such that

$$(37) \quad \frac{1}{\tau_k^{2r}} \int_{B_{\tau_k}} |F_k - \tilde{f}| dV_\omega \leq 2C_4(\delta_k)^{\frac{1}{2}}.$$

Now, since  $F$  and  $\tilde{f}$  are holomorphic, we have

$$(38) \quad \begin{aligned} \int_{Z \cap U_i} |F - f| dV_{Z, \omega} &= \int_{Z \cap U_i} |F - \tilde{f}| dV_{Z, \omega} \leq \frac{1}{\tau_k^{2r}} \int_{B_{\tau_k}} |F - \tilde{f}| dV_{X, \omega} \\ &\leq \frac{1}{\tau_k^{2r}} \int_{B_{\tau_k}} |F_k - \tilde{f}| dV_{X, \omega} + \frac{1}{\tau_k^{2r}} \int_{B_{\tau_k}} |F - F_k| dV_{X, \omega}. \end{aligned}$$

By (37), we have  $\lim_{k \rightarrow +\infty} \frac{1}{\tau_k^{2r}} \int_{B_{\tau_k}} |F_k - \tilde{f}| dV_{X, \omega} = 0$ . Using (32) and (33), we have

$\lim_{k \rightarrow +\infty} \frac{1}{\tau_k^{2r}} \int_{B_{\tau_k}} |F - F_k| dV_{X, \omega} = 0$ . Therefore, the right-hand side of (38) tends to 0. As a consequence  $\int_{Z \cap U_i} |F - f| dV_{Z, \omega} = 0$  and Theorem 1.3 is proved.  $\square$

We complete here the proof of Theorem 1.3 by establishing the Claim 1 in Step 2 and Claim 2 in Step 3.

**Lemma 2.1.** *The claim 1 in Theorem 1.3 is true.*

*Proof.* We use the standard  $L^2$ -estimate to prove the claim. Since  $b_m$  is not smooth, we construct first a smooth approximation of  $b_m$ . Let  $m, k$  be two fixed constants. Set

$$v_\epsilon(t) := \int_{-\infty}^t \int_{-\infty}^{t_1} \frac{1}{1-2\epsilon} \mathbf{1}_{\{-m-1+\epsilon < s < -m-\epsilon\}} * \rho_{\frac{\epsilon}{4}} ds dt_1$$

---

<sup>3</sup>We remind that, for every  $m_k$  fixed,  $\phi_{m_k}$ ,  $\eta_{m_k}$  and  $\lambda_{m_k}$  are bounded functions.

$$- \int_{-\infty}^0 \int_{-\infty}^{t_1} \frac{1}{1-2\epsilon} \mathbf{1}_{\{-m-1+\epsilon < s < -m-\epsilon\}} * \rho_{\frac{\epsilon}{4}} ds dt_1$$

where  $\rho_{\frac{\epsilon}{4}}$  is the kernel of convolution satisfying  $\text{supp}(\rho_{\frac{\epsilon}{4}}) \subset (-\frac{\epsilon}{4}, \frac{\epsilon}{4})$ . It is easy to check that  $v_\epsilon(t)$  is a smooth approximation of  $b_m(t)$ . Set

$$\eta_\epsilon := s(-v_\epsilon \circ \Psi), \quad \phi_\epsilon := u(-v_\epsilon \circ \Psi), \quad \tilde{h}_\epsilon := h_k \cdot e^{-\Psi - \phi_\epsilon}$$

and

$$B_\epsilon := [\eta_\epsilon i\Theta_{\tilde{h}_{m,k}} - i\partial\bar{\partial}\eta_\epsilon - i(\lambda_\epsilon)^{-1}\partial\eta_\epsilon \wedge \bar{\partial}\eta_\epsilon, \Lambda_\omega].$$

Then  $\eta_\epsilon, \phi_\epsilon, B_\epsilon$  tend to  $\eta_m, \phi_m, B_{m,k}$ .

For every form  $\alpha \in C_c^\infty(X, \wedge^{n,1} T_X^* \otimes L)$ , we have <sup>4</sup>

$$(39) \quad \|(\eta_\epsilon + \lambda_\epsilon)^{\frac{1}{2}}(D'')^* \alpha\|_{\tilde{h}_k}^2 + \|(\eta_\epsilon)^{\frac{1}{2}} D'' \alpha\|_{\tilde{h}_k}^2 \geq \langle B_\epsilon \alpha, \alpha \rangle_{\tilde{h}_k}.$$

and

$$(40) \quad \langle (B_\epsilon + \delta_k \eta_\epsilon \text{Id}) \alpha, \alpha \rangle_{\tilde{h}_\epsilon} \geq (v_\epsilon'' \circ \Psi) \langle [\partial\Psi \wedge \bar{\partial}\Psi, \Lambda_\omega] \alpha, \alpha \rangle_{\tilde{h}_\epsilon}$$

By standard  $L^2$ -estimate (cf. appendix), we can find  $\gamma_\epsilon$  and  $\beta_\epsilon$  such that

$$(41) \quad \bar{\partial}\gamma_\epsilon + (\delta_k \eta_\epsilon)^{\frac{1}{2}} \beta_\epsilon = g_{m,k}$$

and

$$(42) \quad \int_X |\gamma_\epsilon|_{\tilde{h}_\epsilon}^2 (\eta_\epsilon + \lambda_\epsilon)^{-1} + \int_X |\beta_\epsilon|_{\tilde{h}_\epsilon}^2 \\ \leq \int_X \{(B_\epsilon + 2\delta_k \cdot \eta_\epsilon(z))^{-1} g_{m,k}, g_{m,k}\}_{\tilde{h}_{\epsilon}} dV_\omega$$

Letting  $\epsilon \rightarrow 0$ , we can find  $\gamma_{m,k}$  and  $\beta_{m,k}$ , such that

$$\bar{\partial}\gamma_{m,k} + (\delta_k \eta_m)^{\frac{1}{2}} \beta_{m,k} = g_{m,k}$$

and

$$(43) \quad \int_X |\gamma_{m,k}|_{\tilde{h}_{m,k}}^2 (\eta_m + \lambda_m)^{-1} + \int_X |\beta_{m,k}|_{\tilde{h}_{m,k}}^2 \\ \leq \int_X \{(B_{m,k} + 2\delta_k \cdot \eta_m(z))^{-1} g_{m,k}, g_{m,k}\}_{\tilde{h}_{m,k}} dV_\omega$$

To finish the proof of the lemma, we rest to check that, for every  $k \in \mathbb{N}$  fixed,

$$(44) \quad \overline{\lim}_{m \rightarrow +\infty} \int_X \{(B_{m,k} + 2\delta_k \eta_m)^{-1} g_{m,k}, g_{m,k}\}_{\tilde{h}_k} dV_{X,\omega} \\ \leq \overline{\lim}_{m \rightarrow +\infty} \int_X (b_m'' \circ \Psi) \cdot \{\tilde{f}_k, \tilde{f}_k\}_{\tilde{h}_{m,k}} dV_\omega$$

By the construction of  $g_{m,k}$  and (40), we have (cf. [Dem12, 12.C])

$$(45) \quad \overline{\lim}_{m \rightarrow +\infty} \int_X \{(B_{m,k} + 2\delta_k \eta_m)^{-1} g_{m,k}, g_{m,k}\}_{\tilde{h}_{m,k}} dV_{X,\omega} \\ \leq \overline{\lim}_{m \rightarrow +\infty} \left( \int_X (b_m'' \circ \Psi) \cdot \{\tilde{f}_k, \tilde{f}_k\}_{\tilde{h}_{m,k}} dV_\omega + \int_X \frac{1 - b_m' \circ \Psi}{\delta_k \cdot \eta_m} \{d'' \tilde{f}_k, d'' \tilde{f}_k\}_{\tilde{h}_{m,k}}^2 dV_\omega \right).$$

Since

$$(1 - b_m' \circ \Psi)(z) = 0 \quad \text{when } \Psi(z) \geq -m,$$

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<sup>4</sup>We refer to [GZ13a, 5.1] for a detailed calculus.

and

$$\eta_m(z) \approx C_1 \cdot |m| \quad \text{when } \Psi(z) \leq -m,$$

we know that

$$\int_X \frac{1 - b'_m \circ \Psi}{\delta_k \cdot \eta_m} \{d'' \tilde{f}_k, d'' \tilde{f}_k\}_{\tilde{h}_{m,k}}^2 dV_\omega \leq \frac{C_1}{\delta_k \cdot |m|} \int_{\Psi \leq -m} |d'' \tilde{f}_k|_{\tilde{h}_{m,k}}^2 dV_\omega.$$

Thanks to the construction of  $\tilde{f}_k$  (i.e., property (iii) in Step 1), by [Li12, Lemma 1.19], for every  $k \in \mathbb{N}$  fixed, we have a key estimate

$$(46) \quad \lim_{m \rightarrow +\infty} \frac{1}{\delta_k \cdot |m|} \int_{\Psi \leq -m} |d'' \tilde{f}_k|_{\tilde{h}_{m,k}}^2 |s|^{-2r} dV_\omega = 0.$$

Combining with the construction of  $\tilde{h}_{m,k}$  and (15), we have

$$\lim_{m \rightarrow +\infty} \frac{1}{\delta_k \cdot |m|} \int_{\Psi \leq -m} |d'' \tilde{f}_k|_{\tilde{h}_{m,k}}^2 dV_\omega = 0.$$

Therefore

$$\overline{\lim}_{m \rightarrow +\infty} \int_X \frac{1 - b'_m \circ \Psi}{\delta_k \cdot \eta_m} \{d'' \tilde{f}_k, d'' \tilde{f}_k\}_{\tilde{h}_{m,k}}^2 dV_\omega = 0$$

Combining with (45), (44) is proved.  $\square$

**Lemma 2.2.** *The claim 2 is true.*

*Proof.* We first recall the Bochner-Martinelli-Koppelman formula.

Let  $D$  be a bounded domain in  $\mathbb{C}^n$  with smooth boundary and  $f \in C_{0,q}^1(\overline{D})$ , the following formula holds

$$(47) \quad f(z) = \int_{\partial D} B_q(\cdot, z) \wedge f + \int_D B_q(\cdot, z) \wedge \bar{\partial}_\xi f + \bar{\partial}_z \int_D B_{q-1}(\cdot, z) \wedge f, z \in D.$$

where

$$\begin{aligned} B(\xi, z) &= \frac{\langle \bar{\xi} - \bar{z}, d\xi \rangle}{|\xi - z|^2} \wedge \left( \frac{\langle d\bar{\xi} - d\bar{z}, d\xi \rangle}{|\xi - z|^2} \right)^{n-1} \\ &= \sum_{q=0}^n B_q, \end{aligned}$$

for  $B_q$  is the summand which is of degree  $(0, q)$  in  $z$ .

We use the formula (47) to construct the  $w_{m_k, k}$  announced in the claim 2. In our context, we first take two open sets  $U'_i$  and  $U''_i$  such that  $U_i \Subset U'_i \Subset U''_i$ . Let  $\theta_i(z)$  be a smooth function with compact support in  $U'_i$  and  $\theta_i(z) = 1$  in a neighborhood of  $U_i$ . During this proof, we set  $\eta^{\frac{1}{2}} := (\eta_{m_k, k})^{\frac{1}{2}}$  and  $\beta := \beta_{m_k, k}$ . Let

$$w_1(z) := \int_{\xi \in U''_i} \theta_i(\xi) \eta^{\frac{1}{2}}(\xi) \beta(\xi) B_0(\xi, z).$$

Applying (47) to  $f = \theta_i(\xi) \eta^{\frac{1}{2}}(\xi) \beta(\xi)$  and  $D = U''_i$ , since  $\theta_i = 0$  near the boundary of  $U''_i$ , we have <sup>5</sup>

$$(48) \quad \theta_i(z) \cdot \eta^{\frac{1}{2}}(z) \cdot \beta(z) = \int_{\xi \in U''_i} B_1(\xi, z) \bar{\partial}(\theta_i(\xi) \eta^{\frac{1}{2}}(\xi) \beta(\xi)) + \bar{\partial} w_1(z).$$

<sup>5</sup>Recall that  $\eta^{\frac{1}{2}}(\xi) \beta(\xi)$  is  $\bar{\partial}$  closed. We should also remark that the  $C^1$ -regularity in the formula is not necessary need in our case, since  $\theta_i(z) \cdot \eta^{\frac{1}{2}}(z) \cdot \beta(z)$  equals to 0 near the boundary and  $\bar{\partial}(\theta_i(\xi) \eta^{\frac{1}{2}}(\xi) \beta(\xi))$  is  $L^1$ . Therefore (48) is still valid in the sense of distributions.

Set  $G(z) := \int_{U_i''} B_1(\xi, z) \bar{\partial}(\theta_i(\xi) \eta^{\frac{1}{2}}(\xi) \beta(\xi))$ . By (48), we know that  $G(z)$  is  $\bar{\partial}$ -closed on  $U_i$ . Moreover, since  $\bar{\partial}(\theta_i(\xi) \eta^{\frac{1}{2}}(\xi) \beta(\xi)) = 0$  in a neighborhood of  $U_i$ , we know that  $G(z)$  is a  $C^\infty$ -function on  $B_{\frac{3a}{4}}$ , and its  $C^m$ -norm can be controlled by:

$$\|G(z)\|_{C^m(B_{\frac{3a}{4}})} \leq C(m) \int_{U_i''} \eta^{\frac{1}{2}}(\xi) \beta(\xi)$$

for every  $m \in \mathbb{N}$ , where  $C(m)$  is a constant depending only on  $m$  and  $a$ . Thanks to (29), the right-hand side of the above inequality is uniformly controlled. By classical elliptic theory, we can find a smooth function  $w_2$  on  $B_{\frac{3a}{4}}$ , such that

$$\bar{\partial}w_2(z) = G(z) \quad \text{for } z \in B_{\frac{a}{2}}$$

together with the estimate

$$(49) \quad \|w_2\|_{C^0(B_{\frac{a}{2}})} \leq C_6.$$

We now prove that  $w_1 + w_2$  satisfies (31). By construction, we have

$$\bar{\partial}(w_1 + w_2)(z) = \eta^{\frac{1}{2}}(z) \beta(z) \quad \text{on } B_{\frac{a}{2}}.$$

Thanks to (49), to prove (31), it is sufficient to prove that

$$(50) \quad \frac{1}{\tau^{2r}} \int_{B_\tau} |w_1| \leq C_7 \quad \text{for every } \tau \leq \frac{1}{2}a.$$

We suppose for simplicity that  $r = 1$ . The general case follows by iterating  $r$  times this procedure. By the definition of  $w_1$ , we have<sup>6</sup>

$$(51) \quad \begin{aligned} \frac{1}{\tau^2} \int_{B_\tau} |w_1| dV_\omega &\leq \frac{1}{\tau^2} \int_{|z_1| \leq \tau; \xi \in U_i''} \frac{|\eta(\xi) \beta(\xi)|}{|\xi - z|^{2n-1}} dV_\xi \wedge dV_z \\ &\leq \frac{1}{\tau^2} \left( \int_{|z_1 - \xi_1| \leq 3\tau; |\xi_1| \leq 2\tau} \frac{|\eta(\xi) \beta(\xi)|}{|\xi - z|^{2n-1}} dV_\xi \wedge dV_z + \int_{|z_1| \leq \tau; |\xi_1| \geq 2\tau} \frac{|\eta(\xi) \beta(\xi)|}{|\xi - z|^{2n-1}} dV_\xi \wedge dV_z \right). \end{aligned}$$

We first calculate the first part of the right-hand side of (51). By a direct calculus, we have

$$(52) \quad \begin{aligned} &\int_{|z_1 - \xi_1| \leq 3\tau} \frac{1}{|z - \xi|^{2n-1}} dV_z \\ &\leq C_7 \tau^2 + \int_{|z_1 - \xi_1| \leq 3\tau} \frac{1}{|z_1 - \xi_1|} d'z_1 \wedge d''z_1 \leq C_8 \tau. \end{aligned}$$

By Cauchy inequality,

$$\int_{|\xi_1| \leq 2\tau} |\eta(\xi) \beta(\xi)| dV_\xi \leq \left( \int_{|\xi_1| \leq 2\tau} |\eta(\xi)|^2 |\xi_1|^2 dV_\xi \right)^{\frac{1}{2}} \left( \int_{|\xi_1| \leq 2\tau} \frac{|\beta|^2}{|\xi_1|^2} dV_\xi \right)^{\frac{1}{2}}$$

Thanks to (25) and (15),

$$(53) \quad \int_X \frac{|\beta|^2}{|\xi_1|^2} dV_\xi \leq C_9.$$

Using (16),

$$\int_{|\xi_1| \leq 2\tau} |\eta(\xi)|^2 |\xi_1|^2 dV_\xi \leq C_{10} \tau^2.$$

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<sup>6</sup>Remind that  $B_0, B_1$  are of order  $\frac{1}{|z - \xi|^{2n-1}}$ .

Therefore

$$(54) \quad \int_{|\xi_1| \leq 2\tau} |\eta(\xi)\beta(\xi)| dV_\xi \leq C_{11}\tau$$

Combining (52) and (54), we have

$$\begin{aligned} & \frac{1}{\tau^2} \left( \int_{|z_1 - \xi_1| \leq 3\tau; |\xi_1| \leq 2\tau} \frac{|\eta(\xi)\beta(\xi)|}{|\xi - z|^{2n-1}} dV_\xi \wedge dV_z \right. \\ &= \frac{1}{\tau^2} \cdot \int_{|\xi_1| \leq 2\tau} |\eta(\xi)\beta(\xi)| dV_\xi \int_{|z_1 - \xi_1| \leq 3\tau} \frac{1}{|z - \xi|^{2n-1}} dV_z \\ &\quad \left. \leq \frac{1}{\tau^2} C_8 \tau \cdot C_{11} \tau = C_8 \cdot C_{11}. \right. \end{aligned}$$

Therefore, the first part of the right-hand side of (51) is uniformly controlled.

As for the second part of the right-hand side of (51), by the same calculus as in (52), we have

$$\frac{1}{\tau^2} \int_{|z_1| \leq \tau, |\xi_1| \geq 2\tau} \frac{|\eta(\xi)\beta(\xi)|}{|\xi - z|^{2n-1}} dV_\xi \wedge dV_z \leq \frac{1}{\tau^2} \int_{|\xi_1| \geq 2\tau} |\eta(\xi)\beta(\xi)| (C_9 \tau^2 + \frac{\tau^2}{|\xi_1|}) dV_\xi$$

(because  $|z_1 - \xi_1| \geq \frac{|\xi_1|}{2}$  in this domain.) The Cauchy inequality implies that

$$\int_{|\xi_1| \geq 2\tau} |\eta(\xi)\beta(\xi)| (C_9 + \frac{1}{|\xi_1|}) dV_\xi \leq \left( \int_{|\xi_1| \geq 2\tau} |\beta(\xi)|^2 (C_9 + \frac{1}{|\xi_1|})^2 dV_\xi \right)^{\frac{1}{2}} \cdot \int_{|\xi_1| \geq 2\tau} |\eta(\xi)|^2 dV_\xi.$$

By (16) and (53), both  $\int_X |\eta|^2 dV_{X,\omega}$  and  $\int_{|\xi_1| \geq 2\tau} |\beta(\xi)|^2 (C_9 + \frac{1}{|\xi_1|})^2 dV_\xi$  are uniformly bounded. Therefore the second part of the right-hand side of (51) is uniformly bounded. In conclusion, (50) is proved.  $\square$

### 3. APPLICATIONS

**3.1. Some direct applications.** As pointed out in Remark 3 in the introduction, Theorem 1.3 implies that

**Corollary 3.1.** *Let  $X$  be a complete Kähler manifold and  $\text{pr} : X \rightarrow \Delta$  be a proper map to a ball  $\Delta \subset \mathbb{C}^1$  centered at 0 of radius  $R$ . Let  $(L, h)$  be a holomorphic line bundle over  $X$  equipped with a hermitian metric (maybe singular)  $h$  such that  $i\Theta_h(L) \geq 0$ . Suppose that  $X_0 := \text{pr}^{-1}(0)$  is a smooth subvariety of codimension 1. Let  $f \in H^0(X_0, K_{X_0} + L)$ . Then there exists a section  $F \in H^0(X, K_X + L)$  such that*

$$(55) \quad \frac{1}{\pi R^2} \int_X \{F, F\}_h \leq \int_{X_0} \{f, f\}_h$$

and  $F|_{X_0} = \text{pr}^*(dt) \wedge f$ , where  $t$  is the standard coordinate of  $\mathbb{C}^1$ .

By the same arguments as in [BP10, A.1], Corollary 3.1 implies the following result :

**Corollary 3.2.** *Let  $X$  be a Kähler manifold and  $\text{pr} : X \rightarrow \Delta$  be a proper map to the ball  $\Delta \subset \mathbb{C}^1$  centered at 0 of radius  $R$ . Let  $(L, \varphi)$  be a holomorphic line bundle over  $X$  equipped with a hermitian metric (maybe singular)  $\varphi$  such that  $i\Theta_\varphi(L) \geq 0$ . Suppose that  $X_0 := \text{pr}^{-1}(0)$  is smooth of codimension 1. Let  $f \in H^0(X_0, mK_{X_0} \otimes L)$ . We suppose that*

$$\int_{X_0} \{f, f\}^{\frac{1}{m}} e^{-\frac{1}{m}\varphi} < +\infty$$

and there exists an extension  $F$  of  $f$  in  $H^0(X, mK_X \otimes L)$  such that

$$\int_X \{F, F\}^{\frac{1}{m}} e^{-\frac{1}{m}\varphi} < +\infty.$$

Then there exists another extension  $\tilde{F}$  of  $f$  in  $H^0(X, mK_X \otimes L)$  such that

$$(56) \quad \frac{1}{\pi R^2} \int_X \{\tilde{F}, \tilde{F}\}^{\frac{1}{m}} e^{-\frac{1}{m}\varphi} \leq \int_{X_0} \{f, f\}^{\frac{1}{m}} e^{-\frac{1}{m}\varphi}$$

and  $\tilde{F}|_{X_0} = \text{pr}^*(dt) \wedge f$ , where  $t$  is the standard coordinate of  $\mathbb{C}^1$ .

*Proof.* The proof given here follows closely [BP10, A.1]. Set

$$C_1 := \int_{X_0} \{f, f\}^{\frac{1}{m}} e^{-\frac{1}{m}\varphi} \quad \text{and} \quad C_2 := \frac{1}{\pi R^2} \int_X \{F, F\}^{\frac{1}{m}} e^{-\frac{1}{m}\varphi}.$$

If  $C_1 \leq C_2$ , then  $F$  satisfies (56) and the corollary is proved. If  $C_1 < C_2$ , since  $F$  is holomorphic, we can apply Corollary 3.1 with weight

$$\varphi_1 := \frac{m-1}{m} \ln |F|^2 + \frac{1}{m} \varphi$$

on the line bundle  $(m-1)K_X \otimes L$ , and obtain a new extension  $F_1$  of  $f$  satisfying

$$(57) \quad \frac{1}{\pi R^2} \int_X \frac{|F_1|^2}{|F|^{\frac{2(m-1)}{m}}} e^{-\frac{1}{m}\varphi} \leq \int_{X_0} \frac{|f|^2}{|f|^{\frac{2(m-1)}{m}}} e^{-\frac{1}{m}\varphi} = \int_{X_0} \{f, f\}^{\frac{1}{m}} e^{-\frac{1}{m}\varphi}.$$

By Hölder's inequality, we have

$$\frac{1}{\pi R^2} \int_X \{F_1, F_1\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}} \leq \left( \frac{1}{\pi R^2} \int_X \{F, F\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}} \right)^{1-\frac{1}{m}} \cdot \left( \frac{1}{\pi R^2} \int_X \frac{|F_1|^2}{|F|^{\frac{2(m-1)}{m}}} e^{-\frac{1}{m}\varphi} \right)^{\frac{1}{m}}.$$

Combining with (57), we have

$$(58) \quad \frac{1}{\pi R^2} \int_X \{F_1, F_1\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}} \leq (C_2)^{1-\frac{1}{m}} (C_1)^{\frac{1}{m}}.$$

We can repeat the same argument with  $F$  replaced by  $F_1$ , etc. We obtain thus a sequence  $\{F_i\}_{i=1}^{+\infty} \subset H^0(X, mK_X \otimes L)$ , and

$$(59) \quad \frac{1}{\pi R^2} \int_X \{F_i, F_i\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}} \leq \left( \frac{1}{\pi R^2} \int_X \{F_{i-1}, F_{i-1}\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}} \right) \cdot (C_1)^{\frac{1}{m}}$$

If there exists an  $i \in \mathbb{N}$  such that  $F_i$  satisfies (56), then Corollary (3.2) is proved. If not, thanks to (59), we have

$$(60) \quad \frac{1}{\pi R^2} \int_X \{F_i, F_i\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}} \searrow C_1.$$

By passing to a subsequence,  $F_i$  tends to a section  $\tilde{F} \in H^0(X, mK_X \otimes L)$ , and  $\tilde{F}|_Z = f$ . By Fatou lemma, (60) implies that

$$\frac{1}{\pi R^2} \int_X \{\tilde{F}, \tilde{F}\}^{\frac{1}{m}} e^{-\frac{1}{m}\varphi} \leq C_1.$$

Corollary 3.2 is proved.  $\square$

**3.2. Positivity of  $m$ -relative Bergman Kernel metric.** We first recall the definition of  $m$ -relative Bergman Kernel metric (cf. [BP10, A.2], [BP08], [Kaw85], [Tsu07]). Let  $p : X \rightarrow Y$  be a surjective map between two smooth manifolds and let  $(L, h_L)$  be a line bundle on  $X$  equipped with a hermitian metric  $h_L$ . Let  $x \in X$  be a point on a smooth fiber of  $p$ . We first define a hermitian metric  $h$  on  $-(mK_{X/Y} + L)_x$  by

$$(61) \quad \|\xi\|_h^2 := \sup \frac{|\tau(x) \cdot \xi|^2}{(\int_{X_{p(x)}} \{\tau, \tau\}^{\frac{1}{m}} e^{-\frac{1}{m} h_L})^m},$$

where the "sup" is taken over all sections  $\tau \in H^0(X_{p(x)}, mK_{X/Y} + L)$ . The  $m$ -relative Bergman Kernel metric  $h_{X/Y}^{(m)}$  on  $mK_X + L$  is defined to be the dual of  $h$ .

Although the construction of the metric  $h_{X/Y}^{(m)}$  is fiberwise and only defined on the smooth fibers, by using the positivity of direct image arguments, [BP10, Thm 0.1] proved that :

**Theorem 3.3.** [BP10, Thm 0.1] *Let  $p : X \rightarrow Y$  be a fibration between two projective manifolds, and let  $L \rightarrow X$  be a line bundle endowed with a metric (maybe singular)  $\varphi_L$  such that  $i\Theta_{\varphi_L}(L) \geq 0$ . Suppose that there exists a generic point  $z \in Y$  and a section  $u \in H^0(X_z, (K_{X/Y})^m \otimes L)$  such that*

$$\int_{X_z} |u|^{\frac{2}{m}} e^{-\frac{\varphi_L}{m}} < +\infty.$$

*Then the line bundle  $(K_{X/Y})^m \otimes L$  admits a metric with positive curvature current. Moreover, this metric equals to the fiberwise  $m$ -Bergman kernel metric (with respect to  $\varphi_L$ ) on the generic fibers of  $p$ .*

In [GZ13a, Cor 3.7], an alternative proof of Theorem 3.3 is given by using the optimal extension proved in [GZ13a, Thm 2.1]. We should remark that, if  $\varphi_L$  has arbitrary singularity, the proof of in [BP10, Thm 0.1] uses the existence of ample line on  $X$ . Therefore the assumption that  $p$  is a projective map is essential in the proof of Theorem 3.3 in [BP10, Thm 0.1]. However, as pointed out by M. Păun, since the optimal extension proved in Corollary 3.2 is without projectivity assumption, we can use Corollary 3.2 to generalise Theorem 3.3 to arbitrary compact Kähler fibrations, by using the same arguments in [GZ13a, Cor 3.7]. For the reader's convenience, we give the proof of this generalization in this subsection.

To begin with, we first prove the following lemma, which uses the recent important result [GZ13b].

**Lemma 3.4.** *Let  $\varphi$  be a psh function on a Stein open set  $U$ . Set:*

$$\mathcal{I}_m(\varphi)_x := \{f \in \mathcal{O}_x \mid \int_{U_x} |f|^{\frac{2}{m}} e^{-\frac{\varphi}{m}} < +\infty\}.$$

*Then  $\mathcal{I}_m(\varphi)$  is a coherent sheaf.*

*Proof.* We first prove the lemma under the assumption that  $\varphi$  has analytic singularities. In this case, Let  $\pi : \tilde{U} \rightarrow U$  be a resolution of singularities of  $\varphi$ , i.e.,  $\varphi \circ \pi$  can be written locally as

$$\varphi \circ \pi = \sum_i a_i \ln(|s_i|) + O(1),$$



where  $s_i$  are holomorphic functions on  $\tilde{U}$  and  $\bigcup_i \text{Div}(s_i)$  is normal crossing. We suppose that  $K_{\tilde{U}} = K_X + \sum_i b_i \cdot E_i$  and  $\sum_i a_i \cdot \text{Div}(s_i) = \sum_i c_i \cdot E_i$ . Let  $k_i$  be the minimal number in  $\mathbb{Z}^+$  such that  $k_i \cdot \frac{2}{m} > \frac{c_i}{m} - 2b_i - 2$ . It is easy to check that  $\mathcal{I}_m(\varphi) = \pi_*(\mathcal{O}(-\sum_i k_i \cdot E_i))$ . Therefore  $\mathcal{I}_m(\varphi)$  is a coherent sheaf.

We now prove the lemma for arbitrary psh functions. Thanks to [Dem12, 15.B], we can find a sequence of quasi-psh  $\varphi_k$  with analytic singularities and a sequence  $\delta_k \rightarrow 0^+$ , such that

(i):  $\varphi_k$  decrease to  $\varphi$ .

(ii):  $\int_{\{\frac{\varphi}{m} < \frac{(1+\delta_k)\varphi_k}{m} + a_k\}} e^{-\frac{\varphi}{m}} < +\infty$  (cf. [Dem12, proof of Thm 15.3, Step 2]) for certain constant  $a_k$ .

As a consequence, we have  $\mathcal{I}_m((1+\delta_k)\varphi_k) \subset \mathcal{I}_m(\varphi)$ . Since we proved that

$$\mathcal{I}_m((1+\delta_k)\varphi_k)$$

are coherent, by the Noetherian property of coherent sheaf,  $\bigcup_{k=1}^{+\infty} \mathcal{I}_m((1+\delta_k)\varphi_k)$  is also coherent and

$$\bigcup_{k=1}^{+\infty} \mathcal{I}_m((1+\delta_k)\varphi_k) \subset \mathcal{I}_m(\varphi).$$

To prove the lemma, it is sufficient to prove that for every  $f \in \mathcal{I}_m(\varphi)$ , we can find a  $k \in \mathbb{N}$ , such that  $f \in \mathcal{I}_m((1+\delta_k)\varphi_k)$ .

Let  $f$  be a holomorphic germ of  $(\mathcal{I}_m(\varphi))_x$ . Then

$$\int_{U_x} |f|^2 e^{-\frac{\varphi}{m} - \frac{2(m-1)\ln|f|}{m}} < +\infty,$$

for some neighborhood  $U_x$  of  $x$ . By [GZ13b], there exists some  $\delta > 0$ , such that

$$\int_{U_x} |f|^2 e^{-\frac{(1+\delta)\varphi}{m} - \frac{2(1+\delta)(m-1)\ln|f|}{m}} < +\infty.$$

Replacing  $U_x$  by a smaller neighborhood  $U'_x$  of  $x$ , we have

$$(62) \quad \int_{U'_x} |f|^{\frac{2}{m}} e^{-\frac{(1+\delta)\varphi}{m}} < +\infty.$$

We take a  $k \in \mathbb{N}$ , such that  $\delta_k < \delta$ . Thanks to (i) and (62), we have

$$\int_{U'_x} |f|^{\frac{2}{m}} e^{-\frac{(1+\delta_k)\varphi_k}{m}} < +\infty.$$

Therefore  $f \in \mathcal{I}_m((1+\delta_k)\varphi_k)$  and the lemma is proved.  $\square$

We now generalise [BP10, Thm 0.1] to arbitrary compact Kähler fibrations. The proof is almost the same as [GZ13a, Cor 3.7].

**Theorem 3.5.** *Let  $p : X \rightarrow Y$  be a fibration between two compact Kähler manifolds. Let  $L \rightarrow X$  be a line bundle endowed with a metric (maybe singular)  $\varphi_L$  such that  $i\Theta_{\varphi_L}(L) \geq 0$ . Suppose that there exists a generic point  $z \in Y$  and a section  $u \in H^0(X_z, (K_{X/Y})^m \otimes L)$  such that*

$$\int_{X_z} |u|^{\frac{2}{m}} e^{-\frac{\varphi_L}{m}} < +\infty.$$

Then the line bundle  $(K_{X/Y})^m \otimes L$  admits a metric with positive curvature current. Moreover, this metric equals to the fiberwise  $m$ -Bergman kernel metric (with respect to  $\varphi_L$ ) on the generic fibers of  $p$ .

*Proof.* By Lemma 3.4,  $p_*((K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi_L))$  is coherent. Using [Fle81] (cf. also [BDIP02, Thm 10.7, page 47]), there exists a subvariety  $Z$  of  $Y$  of codimension at least 1 such that  $p$  is smooth on  $Y \setminus Z$  and for every point  $t \in Y \setminus Z$ , we have

$$\dim H^0(X_t, (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi_L)|_{X_t}) = \text{rank } p_*((K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi_L)),$$

where  $\mathcal{I}_m(\varphi_L)|_{X_t}$  is the restriction of the coherent sheaf  $\mathcal{I}_m(\varphi_L)$  on  $X_t$ . By local extension theorem, we know that  $\mathcal{I}_m(\varphi_L)|_{X_t} \subset \mathcal{I}_m(\varphi_L)|_{X_t}$ . As a consequence, for every Stein neighborhood  $U$  of  $t \in Y \setminus Z$ , the fibration  $p : p^{-1}(U) \rightarrow U$  and the point  $t$  satisfy the conditions in Corollary 3.2.

Let  $h^{(m)}$  be the fiberwise  $m$ -Bergman kernel metric on  $p^{-1}(Y \setminus Z) \rightarrow Y \setminus Z$  (cf. construction in the beginning of this subsection). For every  $x \in p^{-1}(Y \setminus Z)$ , we now estimate the curvature of  $h^{(m)}$  near  $x$ . Let  $e$  be a local coordinate of  $(K_{X/Y})^m \otimes L$  near  $x$ . Let

$$(63) \quad B(z) := \sup \frac{|u'(z)|^2}{(\int_{X_{p(z)}} \{u, u\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}})^m},$$

where the "sup" is taken over all sections  $u \in H^0(X_{p(z)}, (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$  and  $u = u' \cdot e$ . Thanks to (61), to prove that the curvature of  $h^{(m)}$  is positive near  $x$ , it is sufficient to prove that  $\ln B(z)$  is psh near  $x$ .

For every fixed point  $z$  near  $x$ , we can find a section  $u_1 \in H^0(X_{p(z)}, (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$  such that

$$B(z) = \frac{|u'_1(z)|^2}{(\int_{X_{p(z)}} \{u_1, u_1\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}})^m}.$$

Let  $\Delta_r$  be a one dimensional radius  $r$  disc in  $Y$  centered at  $p(z)$ , and  $\Delta'_r$  be a one dimensional disc in  $X$  passing through  $z$  and  $p(\Delta'_r) = \Delta_r$ . Thanks to Proposition 3.2, there exists an optimal extension of  $u_1 : \tilde{u}_1 \in H^0(p^{-1}(\Delta_r), (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$ , such that

$$(64) \quad \frac{1}{\pi r^2} \int_{p^{-1}(\Delta_r)} \{\tilde{u}_1, \tilde{u}_1\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}} p^*(d't \wedge d''t) \leq \int_{X_{p(z)}} \{u_1, u_1\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}},$$

where  $t$  is coordinate of  $\Delta_r$ . By the definition of  $B(z)$ , we have

$$\begin{aligned} \frac{1}{\pi r^2} \int_{\Delta'_r} \ln B(x) p^*(d't \wedge d''t) &\geq \frac{1}{\pi r^2} \int_{\Delta'_r} \ln \frac{|\tilde{u}'_1(x)|^2}{(\int_{X_{p(x)}} \{\tilde{u}_1(x), \tilde{u}_1(x)\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}})^m} p^*(d't \wedge d''t) \\ &= \frac{1}{\pi r^2} \int_{\Delta'_r} \ln |\tilde{u}'_1|^2 p^*(d't \wedge d''t) - \frac{m}{\pi r^2} \int_{p^{-1}(\Delta_r)} \{\tilde{u}_1, \tilde{u}_1\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}} p^*(d't \wedge d''t), \end{aligned}$$

where  $\tilde{u}_1 = \tilde{u}'_1 \cdot e$ . Using (64) and the holomorphicity of  $\tilde{u}'_1$ , we obtain

$$\frac{1}{\pi r^2} \int_{\Delta'_r} \ln B(x) p^*(d't \wedge d''t) \geq \ln B(z).$$

Therefore,  $\ln B(x)$  is psh in the horizontal direction. By the convexity of  $\ln |u'(x)|$  and the construction of  $\ln B(x)$ ,  $\ln B(x)$  is also psh in the fiberwise direction. Therefore  $\ln B(x)$  is psh on  $p^{-1}(Y \setminus Z)$  and the curvature of  $h^{(m)}$  is semi-positive on  $p^{-1}(Y \setminus Z)$  (in the sense of currents).

Using the arguments in [BP10, A.2], we now prove that  $h^{(m)}$  can be extended to the whole  $X$ . We first express  $h^{(m)}$  locally as the potential form  $e^{-\varphi_{X/Y}}$ , where  $\varphi_{X/Y}$  is a quasi-psh function outside the subvariety  $p^{-1}(Z)$ . By the standard results in pluripotential theory, to prove that  $h^{(m)}$  can be extended to  $X$ , it is sufficient to prove the existence of a uniform constant  $C$  such that

$$(65) \quad \varphi_{X/Y} \leq C \quad \text{on } X \setminus p^{-1}(Z).$$

Let  $U$  be a small open set in  $X$ . Let  $B$  be the function on  $U \setminus p^{-1}(Z)$  defined by (63). Thanks to (61), to prove (65), it is equivalent to prove that  $B$  is uniformly bounded on  $U \setminus p^{-1}(Z)$ . For every  $z \in U \setminus p^{-1}(Z)$ , we can find a  $u_2 \in H^0(X_{p(z)}, (K_{X/Y})^m \otimes L \otimes \mathcal{I}_m(\varphi))$  such that

$$B(z) = |u'_2(z)|^2 \quad \text{and} \quad \int_{X_{p(z)}} \{u_2, u_2\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}} = 1.$$

Using Proposition 3.2, we can find an extension  $\tilde{u}_2$  of  $u_2$ , such that

$$\int_{p^{-1}(p(U))} \{\tilde{u}_2, \tilde{u}_2\}^{\frac{1}{m}} e^{-\frac{\varphi}{m}} \leq C_U,$$

where the constant  $C_U$  depends only on  $U$ . By mean value inequality, we know that  $|u'_2(z)|$  is controlled by a constant depending only on  $C_U$ . The theorem is thus proved.  $\square$

#### 4. APPENDIX

For the reader's convenience, we give the proof of (41) and (42), which is a rather standard estimate (cf. [Dem12, Prop. 12.4, Remark 12.5], [DP03] or [Li12]).

Set  $g := g_{m,k}$ ,  $\eta := \eta_\epsilon$ ,  $B := B_{\epsilon,k}$  and  $\delta := \delta_k$  for simplicity. Let  $Y_k$  be a subvariety of  $X$  such that  $\varphi_k$  is smooth outside  $Y_k$ . Then there exists a complete Kähler metric  $\omega_1$  on  $X \setminus Y_k$ . Set  $\omega_s := \omega + s\omega_1$ . Then  $\omega_s$  is also a complete Kähler metric on  $X \setminus Y_k$  for every  $s > 0$ .

We apply the twist  $L^2$ -estimate (cf. [Dem12, 12.A, 12.B]) for the line bundle  $(L, \tilde{h}_k)$  on  $(X \setminus Y_k, \omega_s)$ . Thanks to (39) and [GZ13a, Lemma 4.1], for every smooth  $(n, 1)$ -form  $v$  with compact support, we have

$$(66) \quad |\langle g, v \rangle_{\omega_s}|^2 \leq \left( \int_{X \setminus Y_k} \langle (B + 2\delta\eta(z))^{-1} g, g \rangle dV_{\omega_s} \right) \cdot \left( \|(\eta + \lambda)^{\frac{1}{2}} D''^* v\|_{\omega_s}^2 + \int_{X \setminus Y_k} \langle 2\delta\eta(z) v, v \rangle dV_{\omega_s} \right)$$

Set  $H_1 := \|\cdot\|_{L^2}$ , where the  $L^2$ -norm  $\|\cdot\|_{L^2}$  is defined with respect to the metrics  $\omega_s$  and  $(L, \tilde{h}_k)$ . Let  $H_2$  be a Hilbert space where the norm is defined by

$$\|f\|_{H_2}^2 := \int_{X \setminus Y_k} 2\delta\eta(z) \cdot |f|_{\tilde{h}_k}^2 dV_{\omega_s}.$$

By (66) and the Hahn-Banach theorem, we can construct a continuous linear map

$$H_1 \oplus H_2 \rightarrow \mathbb{C},$$

which is an extension of the application

$$((\eta + \lambda)^{\frac{1}{2}} D''^* v, v) \rightarrow \langle g, v \rangle_{\omega_s}.$$

Therefore, there exist  $f$  and  $h$  such that

$$\langle g, v \rangle_{\omega_s} = \langle f, (\eta + \lambda)^{\frac{1}{2}} D''^* v \rangle_{\omega_s} + \langle 2\delta\eta(z)h, v \rangle_{\omega_s}$$

and

$$\|f\|_{\omega_s}^2 + \|(\delta\eta(z))^{\frac{1}{2}} h\|_{\omega_s}^2 \leq \int_X (\langle B + 2\delta\eta(z) \rangle^{-1} g, g) dV_{\omega_s}$$

Let  $\beta := 2(\delta\eta(z))^{\frac{1}{2}} \cdot h$  and  $\gamma := (\eta + \lambda)^{\frac{1}{2}} f$ . Then

$$g = D''(\gamma) + (\delta\eta(z))^{\frac{1}{2}} \beta$$

and

$$\left\| \frac{\gamma}{(\lambda + \eta)^{\frac{1}{2}}} \right\|_{(X \setminus Y_k, \omega_s)}^2 + \|\beta\|_{(X \setminus Y_k, \omega_s)}^2 \leq \int_{X \setminus Y_k} (\langle B + 2\delta\eta(z) \rangle^{-1} g, g) dV_{\omega_s}$$

Then (41) and (42) are proved by letting  $s \rightarrow 0^+$ .

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